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Noether symmetries of a modified model in teleparallel gravity and a new approach for exact solutions

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Abstract In this paper, we present the Noether symmetries of flat FRW spacetime in the context of a new action in teleparallel gravity which we construct based on the $f(R)$ version. This modified action contains a coupling between the scalar field potential and magnetism. Also, we introduce an innovative approach, the beyond Noether symmetry (B.N.S.) approach, for exact solutions which carry more conserved currents than the Noether approach. By data analysis of the exact solutions, obtained from the Noether approach, late-time acceleration and phase crossing are realized, and some deep connections with observational data such as the age of the universe, the present value of the scale factor as well as the state and deceleration parameters are observed. In the B.N.S. approach, we consider the dark energy dominated era.

1 Introduction

In the last decade, one of the big challenges for physicists is the explanation of the essence and mechanism of the acceleration of our universe [1–3], in the present era of the universe, which has been confirmed by some observation data such as supernova type Ia [4,5] baryon acoustic oscillations [6] weak lensing [7] and large scale structure [8]. However, a plausible elucidation for this is commonly done using the model of a very exotic fluid called dark energy, which has a negative pressure. Another well-known possibility is to modify Einstein's general relativity [9], making the action of the theory dependent on a function of the curvature scalar R; in a certain limit of the parameters, the theory reduces to general relativity. This procedure of explaining the accelerated expansion of our universe is known as modified gravity. An alternative, consistently describing the gravitational interaction, is one in which one only acknowledges the torsion of spacetime, thus canceling out any effect of the curvature. This approach is known as teleparallel theory [10,11] which is demonstrably equivalent to general relativity. Teleparallel gravity enables one to say that gravity is not due to curvature, but to torsion.

The choice of the unknown functions, somewhat arbitrary, such as coupling functions and potentials in the equations of motion obtained from the point-like Lagrangian of the extended models has given rise to the objection of fine tuning, the very problem whose solutions have been set out through inflationary theories. Therefore, it is desirable to have a path to derive the potential or at least some criteria for acceptable potentials. One such approach is based on the Noether symmetry and it was recently applied by Capozziello et al. [12–15], de Ritis et al. [16,17], Sanyal et al. [18–25], and others [26–42]. The Noether approach, representing several conserved currents (Noether currents), is not conducive to any solutions while matching all or a portion of them with field equations. The more currents there are the more problems pile up. On the other hand, hidden currents derivable from a continuity equation [43,44] are desirable to be included, but when doing so things get worse due to the abundance of currents. The point is, with or without intervening hidden currents, one is compelled to cross out some in order to obtain a solution. In this paper, we introduce a new approach, the B.N.S. approach, in Sect. 4, which keeps the maximum possible number of conserved currents, including Noether currents, hidden and arbitrary ones.

This paper is organized as follows. In Sect. 2 we introduce the model and extract the point-like Lagrangian. In Sect. 3 we gently present the Noether symmetries, invariants and exact solutions of our model. Moreover, by data analysis, we demonstrate that the observational data corroborate our findings. In Sect. 4 we introduce the B.N.S. approach and study our model with it, especially in the dark energy dominated era. In Sect. 5 we consider the corresponding WDW-equation and, finally, in Sect. 6 we conclude the results.

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2 The model

In some articles such as Refs. [45–48], the gravitational action

$$S = \int d^4x\sqrt{-g}\bigg[\frac{M_{Pl}^2}{2}R + \frac{1}{2}\phi_{,\mu}\phi^{,\mu} - V(\phi) - \frac{1}{4}f^2(\phi)F_{\mu\nu}F^{\mu\nu}\bigg]$$

was investigated, the studies of which led to satisfactory results (inflation, late-time-accelerated expansion, ...). Indeed, this action is the most generic action for a single field inflation. Gauge fields are the main driving force for the inflationary background. It is worth to note that there are several fields, such as the vector fields and the nonlinear electromagnetic fields, which are able to produce the negative pressure effects. In some papers such as Refs. [47,48], the authors used this model, perhaps, to answer the question whether or not this model may describe the late-time-accelerated expansion. Maybe, the main motivations for applying such models are the efforts made to obtain a unified model (with a single scalar field) which describes the stages of cosmic evolution. Anyway, such discussions are beyond the scope of this paper. Now, the T-version (teleparallel theory with T) of this action is considered completely. Hence, we have

$$S = \int d^4xe\bigg[\frac{M_{Pl}^2}{2}T + \frac{1}{2}\phi_{,\mu}\phi^{,\mu} - V(\phi) - \frac{1}{4}f^2(\phi)F_{\mu\nu}F^{\mu\nu}\bigg] \tag{1}$$

where $e = \det(e^i_\nu) = \sqrt{-g}$ with e^i_ν being a vierbein (tetrad) basis, T is the torsion scalar, $\phi_{,\mu}$ stands for the components of the gradient of ϕ and $V(\phi)$ is the scalar field potential. The vector potential $\mathbf{A}$ of electromagnetic theory generates the electromagnetic field tensor via the geometric equation $\mathbf{F} = -(\text{antisymmetric part of } \nabla\mathbf{A})$. Hence, for a given 4-potential A_μ, the field strength of the vector field is defined by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \equiv A_{\nu,\mu} - A_{\mu,\nu}$. As is seen, in the action (1) the gauge kinetic function $f^2(\phi)$ is coupled to the strength tensor $F_{\mu\nu}$.

In the flat FRW line element,

$$ds^2 = dt^2 - a^2(t)\left(dx^2 + dy^2 + dz^2\right), \tag{2}$$

the scalar torsion takes the form $T = -6\dot{a}^2/a^2$ [51,52] where a is the scale factor of the universe which depends on time only, and the dot denotes a derivative with respect to time.

Regarding (2), if we introduce the homogeneous and isotropic vector field as

$$A_\mu = (A_0; A_1, A_2, A_3) = \left(\chi(t); \frac{A(t)}{\sqrt{3}}, \frac{A(t)}{\sqrt{3}}, \frac{A(t)}{\sqrt{3}}\right), \tag{3}$$

then we would have

$$F_{\mu\nu}F^{\mu\nu} = \frac{-2\dot{A}^2}{a^2}. \tag{4}$$

However, one can choose the gauge $A_0 = \chi(t) = 0$, by using the gauge invariance [45]. Our background (FRW) implies $A_1 = A_2 = A_3$, hence we have no worry about changing the direction of the vector field in time. The action (1) can be written in the canonical form i.e. $S = \int dt L(Q, \dot{Q}) + \Sigma_0$,

$$L(Q, \dot{Q}) = -3a\dot{a}^2 + \frac{1}{2}a^3\dot{\phi}^2 + \frac{1}{2}af^2(\phi)\dot{A}^2 - a^3V(\phi), \tag{5}$$

$$\text{surface-term} = \Sigma_0 = 0,$$

where the configuration space is $Q = (a, \phi, A)$ with tangent space $TQ = (a, \phi, A, \dot{a}, \dot{\phi}, \dot{A})$. We set the reduced Planck mass, M_{Pl}, equal to 1.

3 Noether symmetry

In this section, we study the Noether symmetry approach for the action (1). We split this section into two subsections. In Sect. 3.1, we study the general form of the Noether symmetry, which is perceived as a Noether gauge symmetry. However, this terminology is wrong because there is no gauge [48–50]. In Sect. 3.2, we study the spatial Noether symmetry, which is distinguished as the common Noether symmetry.

3.1 Noether symmetry (NS): a general approach

The Euler–Lagrange equations for a dynamical system are

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0,$$

where q_i are the generalized positions in the corresponding configuration space (i.e. $Q = \{q_i\}$). The energy function associated with the Lagrangian is given by

$$E_L = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L.$$

According to the point-like Lagrangian (5), the corresponding Euler–Lagrange equation for the scale factor a becomes

$$3\dot{a}^2 + \frac{3a^2\dot{\phi}^2}{2} + \frac{f^2\dot{A}^2}{2} - 3a^2V + 6a\ddot{a} = 0. \tag{6}$$

For the scalar field ϕ, the Euler–Lagrange equation takes the following form:

$$ff'\dot{A}^2 - a^2V' - 3a\dot{\phi}\dot{a} - a^2\ddot{\phi} = 0, \tag{7}$$

which is the Klein–Gordon equation. The prime indicates the derivative with respect to ϕ. For the vector potential A, the

Euler–Lagrange equation reads

$$\dot{a} f^2 \dot{A} + 2aff' \dot{A}\dot{\phi} + af^2 \ddot{A} = 0. \tag{8}$$

Finally, the Hamiltonian constraint or total energy E_L corresponding to the (0,0)-Einstein equation becomes

$$\frac{a^2\dot{\phi}^2}{2} + \frac{f^2\dot{A}^2}{2} + a^2 V - 3\dot{a}^2 = 0. \tag{9}$$

The dynamics of our system is given by these four equations.

The pressure and energy density of the scalar field can be written as

$$\begin{aligned} P_\phi &= \frac{\dot{\phi}^2}{2} - V(\phi) + \frac{f^2(\phi)\dot{A}^2}{6a^2}, \\ \rho_\phi &= \frac{\dot{\phi}^2}{2} + V(\phi) + \frac{f^2(\phi)\dot{A}^2}{2a^2}. \end{aligned} \tag{10}$$

Consequently, the EoS parameter for a scalar field could be

$$W_{\text{eff}} = \frac{P_\phi}{\rho_\phi} = \frac{\frac{\dot{\phi}^2}{2} - V(\phi) + \frac{f^2(\phi)\dot{A}^2}{6a^2}}{\frac{\dot{\phi}^2}{2} + V(\phi) + \frac{f^2(\phi)\dot{A}^2}{2a^2}}. \tag{11}$$

To solve the field Eqs. (6)–(8) we use the Noether symmetry approach.

A vector field

$$\begin{aligned} \mathbf{X} &= \xi(t,a,\phi,A)\frac{\partial}{\partial t} + \alpha(t,a,\phi,A)\frac{\partial}{\partial a} \\ &\quad + \beta(t,a,\phi,A)\frac{\partial}{\partial \phi} + \gamma(t,a,\phi,A)\frac{\partial}{\partial A}, \end{aligned} \tag{12}$$

is a Noether symmetry of the Lagrangian (5), if there exists a vector valued function, $G(t,a,\phi,A) \in \Gamma$, where Γ is the space of differential functions such that

$$\mathbf{X}^{[1]} L + L D_t \xi(t,a,\phi,A) = D_t G(t,a,\phi,A), \tag{13}$$

in which D_t given by

$$D_t \equiv \frac{\partial}{\partial t} + \dot{a}\frac{\partial}{\partial a} + \dot{\phi}\frac{\partial}{\partial \phi} + \dot{A}\frac{\partial}{\partial A}, \tag{14}$$

the total derivative operator, and $\mathbf{X}^{[1]}$, the first-order prolongation is defined by

$$\begin{aligned} \mathbf{X}^{[1]} &= \mathbf{X} + (D_t\alpha - \dot{a} D_t \xi)\frac{\partial}{\partial \dot{a}} + (D_t\beta - \dot{\phi} D_t\xi)\frac{\partial}{\partial \dot{\phi}} \\ &\quad + (D_t\gamma - \dot{A} D_t\xi)\frac{\partial}{\partial \dot{A}}. \end{aligned} \tag{15}$$

If $\mathbf{X}$ is the Noether symmetry corresponding to the Lagrangian (5), then

$$\mathbf{I} = \xi L + (\alpha - \dot{a}\xi)\frac{\partial L}{\partial \dot{a}} + (\beta - \dot{\phi}\xi)\frac{\partial L}{\partial \dot{\phi}} + (\gamma - \dot{A}\xi)\frac{\partial L}{\partial \dot{A}} - G \tag{16}$$

is a conserved quantity associated with $\mathbf{X}$. The Noether symmetry condition for the Lagrangian (5) yields the following system of linear partial differential equations:

$$\xi_a = \xi_\phi = \xi_A = 0, \tag{17}$$
$$G = -3\alpha a^2 V - a^3 V \xi_t - \beta a^3 V', \tag{18}$$
$$G_a = -6a\alpha_t, \quad G_b = a^3\beta_t, \quad G_A = af^2\gamma_t, \tag{19}$$
$$3\alpha + 2a\beta_\phi - a\xi_t = 0, \tag{20}$$
$$\alpha f + 2\beta a f' + 2af\gamma_A - af\xi_t, \tag{21}$$
$$-2\alpha_a + a\xi_t = 0, \quad -6\alpha_\phi + a^2\beta_a = 0, \tag{22}$$
$$f^2\gamma_a - 6\alpha_A = 0, \quad f^2\gamma_a + a^2\beta_A = 0. \tag{23}$$

3.1.1 Zero G

Among the many sets of answers which we found, only $V(\phi)$ and $f(\phi)$ being non-zero are noteworthy. Our solutions are

$$\begin{aligned} \beta &= \frac{-2\sqrt{6}}{3}\left(\frac{c_3 e^{-\sqrt{6}\phi/4}}{a^{3/2}} - c_1\right), \\ \gamma &= \frac{2c_1}{3}A + c_5, \quad \alpha = \frac{-2}{3}\left(\frac{c_3 e^{-\sqrt{6}\phi/4}}{a^{1/2}}\right) + \frac{c_1}{3}a, \\ \xi &= c_1 t + c_2, V(\phi) = V_0 e^{-\sqrt{6}\phi/2}, \quad f(\phi) = f_0 e^{-\sqrt{6}\phi/12}. \end{aligned} \tag{24}$$

Thus, the symmetry generators, X_i, on the tangent space turn out to be

$$\begin{aligned} \mathbf{X}_1 &= \frac{2\sqrt{6}}{3}t\frac{\partial}{\partial t} + \frac{2A}{3}\frac{\partial}{\partial A} + \frac{a}{3}\frac{\partial}{\partial a}, \quad \mathbf{X}_2 = \frac{\partial}{\partial t}, \\ \mathbf{X}_3 &= \frac{-2\sqrt{6}}{3}\frac{e^{-\sqrt{6}\phi/4}}{a^{3/2}}\frac{\partial}{\partial \phi} - \frac{2}{3}\frac{e^{-\sqrt{6}\phi/4}}{\sqrt{a}}\frac{\partial}{\partial a}. \quad \mathbf{X}_4 = \frac{\partial}{\partial A}. \end{aligned} \tag{25}$$

So, the corresponding conserved currents are

$$\begin{aligned} \mathbf{I}_1 &= \frac{-ta^3\dot{\phi}^2}{2} + \frac{taf^2\dot{A}^2}{2} - ta^3 V + 3ta\dot{a}^2 - 2a^2\dot{a} \\ &\quad + \frac{2\sqrt{6}a^3\dot{\phi}}{3} + \frac{2af^2A\dot{A}}{3}, \\ \mathbf{I}_2 &= 3a\dot{a}^2 - \frac{a^3\dot{\phi}^2}{2} + \frac{af^2\dot{A}^2}{2} - a^3 V, \\ \mathbf{I}_3 &= 4e^{-\sqrt{6}\phi/4}\sqrt{a}\dot{a} - \frac{2\sqrt{6}}{3}e^{-\sqrt{6}\phi/4}a^{3/2}\dot{\phi}, \quad \mathbf{I}_4 = af^2\dot{A}. \end{aligned} \tag{26}$$

Now, we use cyclic variables associated with the Noether symmetry generator $\mathbf{X}_3$ to simplify the system of equations. Note that $(\partial \mathbf{I}_4/\partial t) \equiv$ Eq. (8). The existence of the Noether symmetry ensures the presence of cyclic variables, say

$$(t,a,\phi,A) \to (s,w,u,v),$$

such that the Lagrangian becomes cyclic in one of them (w in our case). Cyclic variables can be found by defining a transformation $i : (t,a,\phi,A) \to (s,w,u,v)$ as an interior product such that $i_{\mathbf{X}_3}\mathrm{d}s = 0, i_{\mathbf{X}_3}\mathrm{d}w = 1, i_{\mathbf{X}_3}\mathrm{d}u = 0$ and $i_{\mathbf{X}_3}\mathrm{d}v = 0$ or, put differently, we have the following equations:

$$\begin{aligned}
&\xi\frac{\partial s(t,a,\phi,A)}{\partial t}+\alpha\frac{\partial s(t,a,\phi,A)}{\partial a}\\
&\quad+\beta\frac{\partial s(t,a,\phi,A)}{\partial \phi}+\gamma\frac{\partial s(t,a,\phi,A)}{\partial A}=0,\\
&\xi\frac{\partial w(t,a,\phi,A)}{\partial t}+\alpha\frac{\partial w(t,a,\phi,A)}{\partial a}\\
&\quad+\beta\frac{\partial w(t,a,\phi,A)}{\partial \phi}+\gamma\frac{\partial w(t,a,\phi,A)}{\partial A}=1,\\
&\xi\frac{\partial u(t,a,\phi,A)}{\partial t}+\alpha\frac{\partial u(t,a,\phi,A)}{\partial a}\\
&\quad+\beta\frac{\partial u(t,a,\phi,A)}{\partial \phi}+\gamma\frac{\partial u(t,a,\phi,A)}{\partial A}=0,\\
&\xi\frac{\partial v(t,a,\phi,A)}{\partial t}+\alpha\frac{\partial v(t,a,\phi,A)}{\partial a}\\
&\quad+\beta\frac{\partial v(t,a,\phi,A)}{\partial \phi}+\gamma\frac{\partial v(t,a,\phi,A)}{\partial A}=0,
\end{aligned} \tag{27}$$

where (according to our study i.e. $\mathbf{X}_3$)

$$\alpha=\frac{-2}{3}a^{-1/2}e^{-\sqrt{6}\phi/4},\quad \beta=\frac{-2\sqrt{6}}{3}a^{-3/2}e^{-\sqrt{6}\phi/4},\\ \gamma=\xi=0, \tag{28}$$

i.e. $c_1=c_2=c_5=0$. The coordinate transformation (27) is not unique and a clever choice can be very advantageous. Moreover, the solution of Eq. (27) is, in general, not defined on the entire space but only locally. Among the many sets of solutions which we found, we choose these solutions:

$$s=t,\quad w=\frac{1}{2}a^{3/2}e^{-\sqrt{6}\phi/4},\\ u=\frac{1}{2}a^{3/2}e^{\sqrt{6}\phi/4},\quad v=A. \tag{29}$$

Therefore we have

$$(t,a,\phi,A)\longrightarrow\left(t,\frac{1}{2}a^{3/2}e^{-\sqrt{6}\phi/4},\frac{1}{2}a^{3/2}e^{\sqrt{6}\phi/4},A\right),$$

in which w is a cyclic variable. Thus, the scale factor, scalar field, coupling function, and the scalar field potential can be written as

$$a=(4uw)^{1/3},\quad \phi=\frac{2}{\sqrt{6}}\ln\left(\frac{w}{u}\right),\\ f(\phi)=f_0\left(\frac{u}{w}\right)^{1/6},\quad V(\phi)=\frac{V_0u}{w}. \tag{30}$$

The point-like Lagrangian (5) in terms of the new variables then reads

$$\begin{aligned}\mathcal{L}&=\mathcal{L}(u,w,A,\dot u,\dot w,\dot A)\\ &=\frac{-16}{3}\dot u\dot w-4V_0u^2+2^{-1/3}f_0^2u^{2/3}\dot A^2.\end{aligned} \tag{31}$$

The Euler–Lagrange equations lead to

$$\ddot u=0,\quad 2^{2/3}f_0^2\dot A^2-24V_0u^{4/3}+16\ddot w u^{1/3}=0,\\ 3u\ddot A+2\dot A\dot u=0, \tag{32}$$

and the corresponding conserved current ($\mathbf{I}_3$) will be

$$\tilde{\mathbf{I}}_3=\frac{16}{3}\dot u\qquad\longrightarrow\qquad \ddot u=0. \tag{33}$$

As we observe, this equation does not add any new equation. Solutions for Eq. (32) are

$$\begin{aligned}
u(t)&=c_5t+c_6,\\
A(t)&=c_3+c_4\int\frac{\mathrm{d}t}{u^{2/3}}=c_3+\frac{3c_4(c_5t+c_6)^{1/3}}{c_5}+c_7,\\
w(t)&=\int\left[\int\left(\frac{24V_0u^{4/3}-2^{2/3}f_0^2\dot A^2}{16u^{1/3}}\right)\mathrm{d}t\right]\mathrm{d}t+c_1t+c_2\\
&=\frac{1}{32c_5^2}\left[(2^{1/3}3c_4f_0)^2(c_5t+c_6)^{1/3}+8V_0(c_5t+c_6)^3\right]\\
&\quad+(c_8+c_1)t+c_2+c_9,
\end{aligned} \tag{34}$$

where $\{c_i, i=1,\ldots,9\}$, are constants of integration. Inserting values of $u(t)$, $A(t)$, and $w(t)$ in Eq. (30), we obtain

$$\begin{aligned}a(t)&=\Bigg[\left(\frac{3f_0c_4}{2^{7/6}c_5}\right)^2(c_5t+c_6)^{4/3}+\frac{V_0}{c_5^2}(c_5t+c_6)^4\\ &\quad+4(c_5t+c_6)\{(c_1+c_8)t+c_2+c_9\}\Bigg]^{1/3},\end{aligned} \tag{35}$$

$$\begin{aligned}\phi(t)&=\frac{\sqrt{6}}{3}\ln\Bigg[\left(\frac{3f_0c_4}{2^{13/6}c_5}\right)^2(c_5t+c_6)^{-2/3}+\frac{V_0}{4c_5^2}\\ &\quad\times(c_5t+c_6)^2+(c_5t+c_6)^{-1}\{(c_1+c_8)t+c_2+c_9\}\Bigg],\end{aligned} \tag{36}$$

$$\begin{aligned}f(\phi)&=f_0\exp\left[\frac{\sqrt{6}\phi}{-12}\right]=f_0N(\phi)\\
&\equiv f_0\Bigg[(32c_5^2(c_5t+c_6))\Big((2^{1/3}3c_4f_0)^2(c_5t+c_6)^{1/3}\\
&\quad+8V_0(c_5t+c_6)^3+32c_5^2(c_8+c_1)t\\
&\quad+32c_5^2(c_2+c_9)\Big)^{-1}\Bigg]^{1/6}=f(t),
\end{aligned} \tag{37}$$

$$\begin{aligned}V(\phi)&=V_0\exp\left[\frac{\sqrt{6}\phi}{-2}\right]\\
&\equiv V_0\Bigg[(32c_5^2(c_5t+c_6))\Big((2^{1/3}3c_4f_0)^2(c_5t+c_6)^{1/3}\\
&\quad+8V_0(c_5t+c_6)^3+32c_5^2(c_8+c_1)t\\
&\quad+32c_5^2(c_2+c_9)\Big)^{-1}\Bigg]=V(t).
\end{aligned} \tag{38}$$

To see the behavior of important quantities with these solutions, we choose constants as follows:

$$\begin{aligned}
&c_1 = 0.2795084975, \quad c_2 = 1.615579240 \times 10^{10},\\
&c_4 = 1.120082910 \times 10^{-14},\\
&c_5 = 1.118033986 \times 10^{-21},\\
&f_0 = \sqrt{-1} = i, \quad V_0 = 10.00000003,\\
&c_3 = c_6 = c_7 = c_8 = c_9 = 0.
\end{aligned} \tag{39}$$

Perhaps, the value of f_0 seems strange, but the square of f_0 matters in the action (1) and also in the relevant equations, so it is not problematic.

We present four figures with a data analysis. Figure 1a indicates the scale factor, of an increasing character, expressing first the decelerated and then the accelerated expansion of the universe. The present values of the age of the universe and the scale factor are $t_0 = 13.80$ Gyr and $a_0 = 1.00$, respectively. Figure 1b indicates that the redshift goes down, while the scale factor increases with time. The present value of the redshift is $z_0 = 0$. Figure 2a, b show the scalar field and the Hubble parameter with decreasing natures versus time, as we expect. The present values of the Hubble parameter and scalar field are $H_0 = 5.7212 \times 10^{-11}\, yr^{-1} \equiv 55.98\,\text{km.s}^{-1}.\text{Mps}^{-1}$ and $\phi_0 = 39.95$. We do not present plots of the scalar potential $V(\phi)$ and the coupling function $f^2(\phi)$ versus ϕ because their behaviors are obvious (decreasing exponentially). In Fig. 3 $V(\phi)$ and $f^2(\phi)$ are plotted with respect to time. Figure 3a shows the scalar potential increases with time, while Fig. 3b indicates the coupling function $f^2(\phi)$ decreasing with time; however, the absolute value of $f^2(\phi)$ is increasing such as $V(\phi)$ vs. time. Astrophysical data show that W_{eff} lies in a very narrow band close to $W_{\text{eff}} = -1$. The behavior of W_{eff} in Fig. 4a indicates that the crossing of the phantom divide line $W_{\text{eff}} = -1$ occurs from the quintessence phase $W_{\text{eff}} > -1$ to the phantom phase $W_{\text{eff}} < -1$. The present value of the EoS parameter is calculated to be $W_{\text{eff}_0} = -1.00$. The deceleration parameter, $q = -(a\ddot{a})/\dot{a}^2$, shows first a positive, indicative of decelerating universe, and then a negative behavior, implying an accelerating universe (see Fig. 4b). So, late-time-accelerated expansion is realized. The present value of the

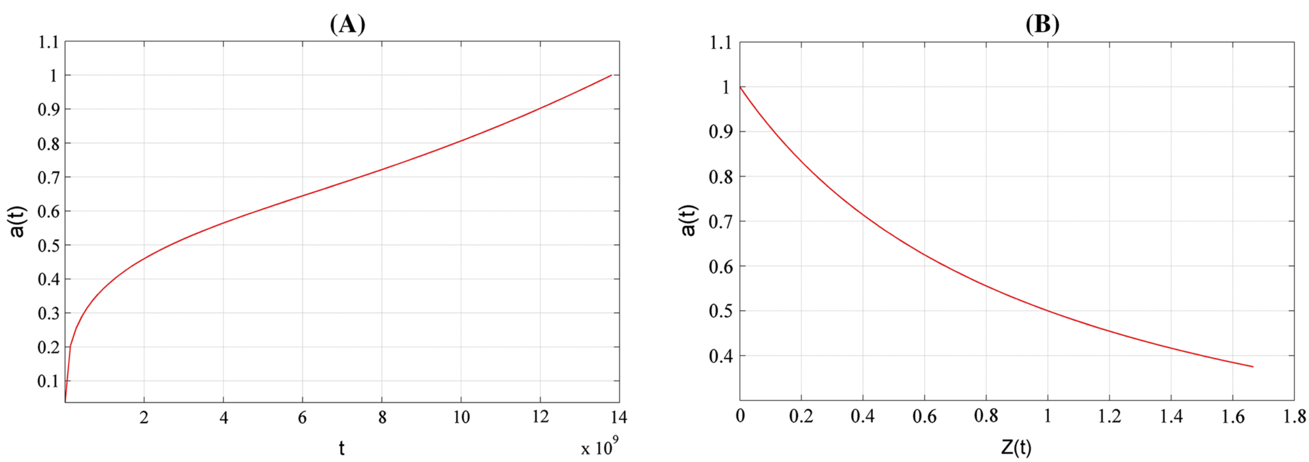

Fig. 1 *Plot* **a** indicates the scale factor $a(t)$ versus time t at the time range [0.7 Myr, 13.8 Gyr] while **b** shows the scale factor $a(t)$ versus redshift z at the time range [1 Gyr, 13.8 Gyr]

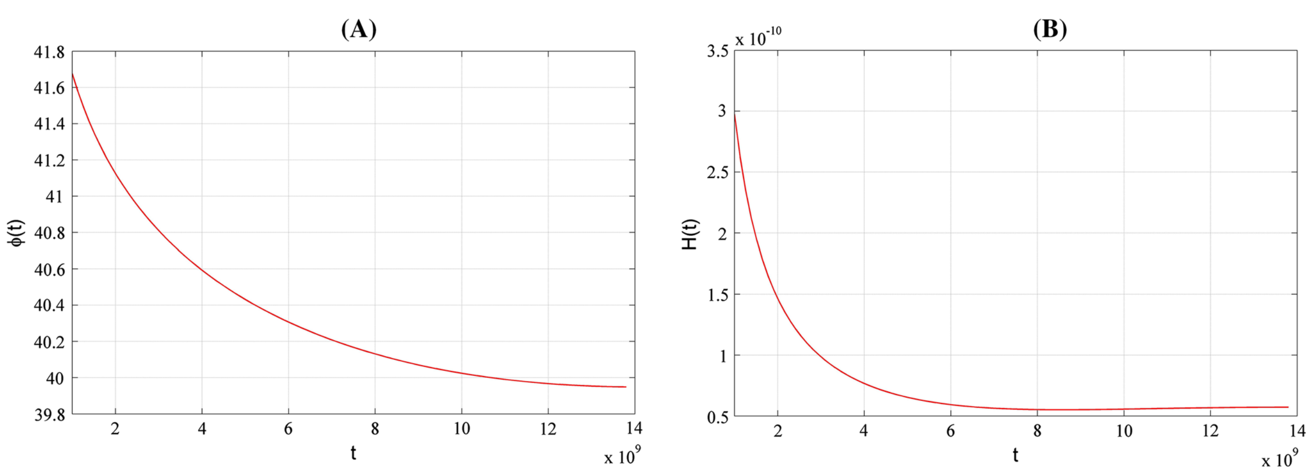

Fig. 2 *Plots* **a** and **b** indicate the scalar field $\phi(t)$ and the Hubble parameter $H(t)$ versus time t at the time range [1 Gyr, 13.8 Gyr], respectively

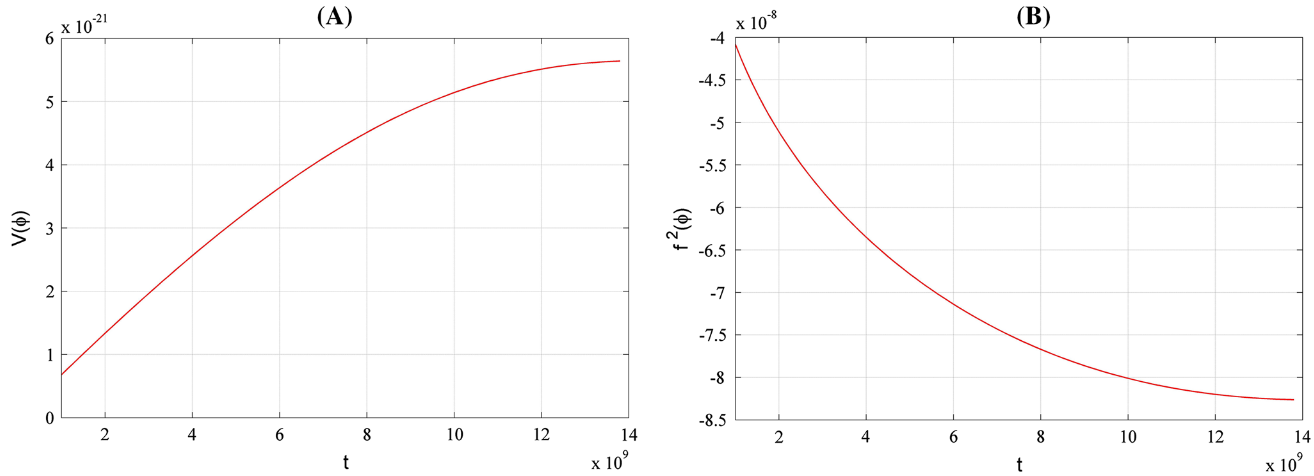

Fig. 3 *Plots* **a** and **b** indicate the scalar potential $V(\phi)$ and coupling function $f^2(\phi)$ versus time t at the time range [1 Gyr, 13.8 Gyr], respectively

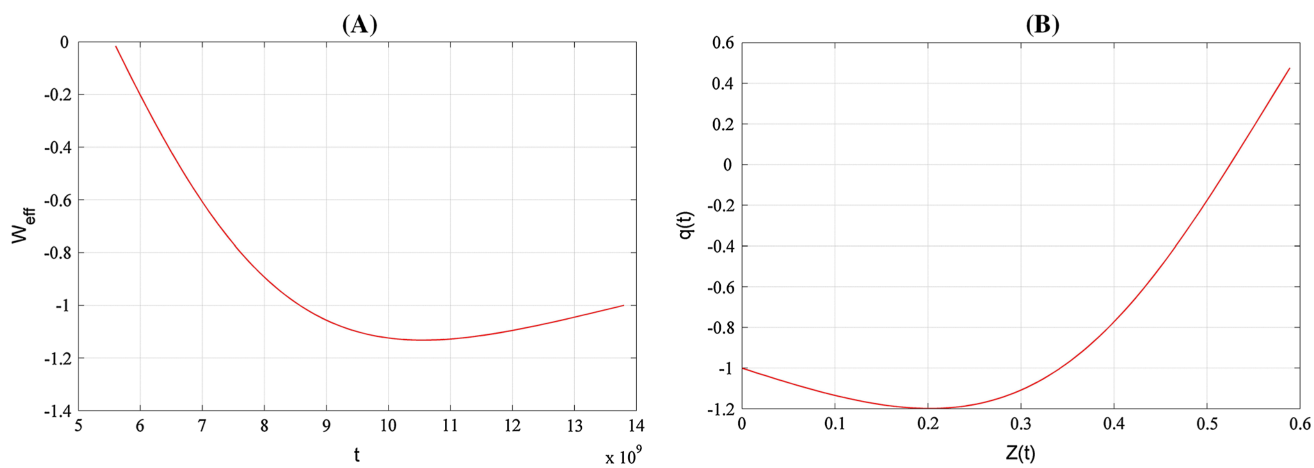

Fig. 4 *Plots* **a** and **b** indicate the state parameter $W_{\rm eff}$ and deceleration parameter q versus redshift z at the time range [5.6 Gyr, 13.8 Gyr], respectively

deceleration parameter is measured to be $q_0 = -1.00$, and at the time $t_{ac.} = 6.294\,Gyr$, we have $q(t_{ac.}) = 0$, so the acceleration starts at the redshift value $z(t_{ac.}) = 0.524$, which is at about half the age of the universe.

- Satisfaction of Maxwell's equations

Here, we want to answer the question whether, with the obtained form of the vector potential, the Maxwell's equations are satisfied. For this purpose, we must utilize Maxwell's equations in curved spacetime, which in terms of the components of the field tensor **F** are [54]

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0, \tag{40}$$

$$F^{\alpha\beta}{}_{,\beta} = -4\pi J^{\alpha}; \quad \begin{cases} \text{if } \alpha = 0: \; J^0 = \rho = \text{charge density}, \\ \text{if } \alpha \neq 0: \; (J^1, J^2, J^3) \\ \qquad\qquad = \text{components of current density}, \end{cases} \tag{41}$$

where $\{J^{\alpha};\, \alpha \in \{0, 1, 2, 3\}\}$ are the components of the 4-current **J**. In a nutshell, through Eq. (40) as regards magnetodynamics and magnetostatics, and through Eq. (41) as regards electrodynamics and electrostatics, we have unification in one geometric law. The usual form of Maxwell's equations may be obtained easily since Eq. (40) reduces to $\nabla\cdot\mathbf{B} = 0$ when one takes $\alpha = 1$, $\beta = 2$, $\gamma = 3$; and it reduces to $\partial\mathbf{B}/\partial t + \nabla \times \mathbf{E} = 0$ when one sets any index, e.g., $\alpha = 0$, and finally, with Eq. (41) two of Maxwell's equations, $\nabla \cdot \mathbf{E} = 4\pi\rho$ (the electrostatic equation), $\partial\mathbf{E}/\partial t - \nabla\times\mathbf{B} = -4\pi\mathbf{J}$ (the electrodynamic equation), are obtained by putting $\alpha = 0$ and $\alpha \neq 0$, respectively.

For the electromagnetism part of the action (1), i.e.

$$\mathcal{L}_{EM} = -\frac{1}{4}\int d^4x\sqrt{-g}\,f^2(\phi)F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}\int d^4x\sqrt{-g}\,g^{\alpha\beta}g^{\mu\nu}f^2(\phi)F_{\mu\alpha}F_{\nu\beta}, \tag{42}$$

Eqs. (40) and (41) read

$$\partial^{\alpha}\left(\sqrt{-g}f^{2}(\phi)F^{\beta\gamma}\right)+\partial^{\beta}\left(\sqrt{-g}f^{2}(\phi)F^{\gamma\alpha}\right)+\partial^{\gamma}\left(\sqrt{-g}f^{2}(\phi)F^{\alpha\beta}\right)=0, \tag{43}$$

$$\partial_{\mu}\left[\sqrt{-g}g^{\alpha\beta}g^{\mu\nu}f^{2}(\phi)F_{\nu\beta}\right]=0 \longrightarrow \left(\sqrt{-g}f^{2}(\phi)F^{\alpha\mu}\right)_{,\mu}=0, \tag{44}$$

respectively. Note that in our case, we have $\mathbf{J}=(J^0, J^1, J^2, J^3)=(0,0,0,0)$. After simplifying, Eqs. (43) and (44) lead to the same equation, viz.

$$\frac{\partial}{\partial t}\left(af^{2}\dot{A}\right)=0. \tag{45}$$

Clearly, this equation is equivalent to the third field equation (i.e. Eq. 8). Hence, Eqs. (40) and (41) are satisfied automatically when the solution for the field equations was found. Therefore, the results are consistent with all Maxwell's field equations.

Now, let us define the electric $\mathbf{E}$ and magnetic $\mathbf{B}$ fields covariantly, which are seen by an observer who is characterized by the 4-velocity vector u^{μ}. One has [55]

$$E_{\mu}=u^{\nu}F_{\mu\nu},\quad B_{\mu}=\frac{1}{2}\varepsilon_{\mu\nu\kappa}F^{\nu\kappa}, \tag{46}$$

where the tensor $\varepsilon_{\mu\nu\kappa}$ is defined by the relation

$$\varepsilon_{\mu\nu\kappa}=\eta_{\mu\nu\kappa\lambda}u^{\lambda}, \tag{47}$$

in which $\eta_{\mu\nu\kappa\lambda}$ is an antisymmetric permutation tensor of spacetime with $\eta^{0123}=1/\sqrt{-g}$ or $\eta_{0123}=\sqrt{-g}$. In cosmic time for a comoving observer with $u^{\mu}=(1,0,0,0)$, we get

$$E_{\mu}=\begin{cases}E_i=-\dot{A}_i; & \text{for } \mu=i=1,2,3\\ 0; & \text{for } \mu=0\end{cases},\qquad B_{\mu}=\begin{cases}B_i=\frac{1}{a}\epsilon_{ijk}\partial_j A_k; & \text{for } \mu=i,j,k\in\{1,2,3\}\\ 0; & \text{otherwise,}\end{cases} \tag{48}$$

where ϵ_{ijk} is the well-known Levi-Civita symbol with $\epsilon_{123}=1$. In our case, we have obtained the form $A_{\mu}=(\chi(t),\theta t^{1/3},\theta t^{1/3},\theta t^{1/3})$ with $\theta=\sqrt{3}c_4c_5^{-2/3}$ for the 4-vector potential, hence we get

$$E_{\mu}=\frac{\theta}{3}t^{-2/3}(0,1,1,1),\quad B_{\mu}=\frac{\theta}{3a}t^{-2/3}(0,1,1,1). \tag{49}$$

According to these forms, both E and B decay with time. Data analysis shows that the present values are $E_{\mu 0}=3.13\times10^{-7}(0,1,1,1)$ and $B_{\mu 0}=3.13\times10^{-7}(0,1,1,1)$, so their norms at present time are equal, i.e., $\|\mathbf{E}\|_0=\|\mathbf{B}\|_0$.

3.1.2 Non-zero G

Because in our study we focus on the non-constant form of $V(\phi)$ and $f(\phi)$, this would lead to strange results for any choice of the function $G(t,a,\phi,A)$ except zero. Therefore, by a non-zero G-function, we will not have any conserved current, because all the Noether coefficients are zero.

3.2 Spatial Noether symmetry (SNS)

Obviously, for getting the SNS-equations, we must take $G=\xi=0$. Our solutions for these equations are

$$\begin{aligned}\beta&=\frac{c_1c_4\sqrt{6}\left(c_2e^{\sqrt{6}\phi/4}-c_3e^{-\sqrt{6}\phi/4}\right)}{a^{3/2}},\quad \gamma=c_8,\\ \alpha&=\frac{c_1c_4\left(c_2e^{\sqrt{6}\phi/4}+c_3e^{-\sqrt{6}\phi/4}\right)}{a^{1/2}},\\ f(\phi)&=c_9e^{\sqrt{6}\phi/12}\left(c_2-c_3e^{-\sqrt{6}\phi/2}\right)^{1/3},\\ V(\phi)&=c_7\left(c_3^2e^{-\sqrt{6}\phi/2}-2c_2c_3+c_2^2e^{\sqrt{6}\phi/2}\right).\end{aligned} \tag{50}$$

The symmetry generators, X_i, on the tangent space, become

$$\begin{aligned}\mathbf{X}_1&=\sqrt{\frac{6}{a}}e^{\sqrt{6}\phi/4}\left(\frac{\partial}{\partial a}+\frac{1}{a}\frac{\partial}{\partial\phi}\right),\\ \mathbf{X}_2&=\sqrt{\frac{6}{a}}e^{-\sqrt{6}\phi/4}\left(\frac{\partial}{\partial a}-\frac{1}{a}\frac{\partial}{\partial\phi}\right),\quad \mathbf{X}_3=\frac{\partial}{\partial A}.\end{aligned} \tag{51}$$

Consequently, the corresponding conserved currents take the forms

$$\begin{aligned}\mathbf{I}_1&=\sqrt{6a}e^{\sqrt{6}\phi/4}\left(-\sqrt{6}\dot{a}+a\dot{\phi}\right),\\ \mathbf{I}_2&=\sqrt{6a}e^{-\sqrt{6}\phi/4}\left(-\sqrt{6}\dot{a}-a\dot{\phi}\right),\qquad \mathbf{I}_3=af^2\dot{A}.\end{aligned} \tag{52}$$

By some subtle moves, one can find that this case is similar to the NS case (i.e. the results will be the same). So, it is futile to follow this set of solutions. It is sufficient to say that these generators (and also the conserved currents) correspond to $\mathbf{X}_3$ and $\mathbf{X}_4$ (so $\mathbf{I}_3$ and $\mathbf{I}_4$) in the NS approach (see Eqs. 25 and 26).

4 Beyond Noether symmetry approach (B.N.S. approach)

In this section, we have an novel approach for exact solutions. We named it the "B.N.S. approach" (for "beyond Noether symmetry approach"). This approach is useful for extended gravity because we have some degrees of freedom. Also, we may have more conserved currents. Let us explain this approach.

In any action of extended gravity, we have some unknown functions, such as $V(\phi)$ and $f(\phi)$ in our case. Regularly, we may use the Noether approach for defining them. As we observe in our case, and also in almost all other cases in the literature, we cannot obtain the solutions which carry all conserved currents or at least more of those. For solving the field equations, we have to remove some of the conserved currents. On the other hand, symmetries have always played a central role in the conceptual discussion of classical and quantum physics. We found that the main problem is the form of these unknown functions, the main culprit in removing some of the conserved currents. In the case in which we have new forms of these unknown functions, the problem can be solved. The B.N.S. approach carries it out in a simple way. Suppose that $F_1(\varphi), F_2(\varphi), \ldots, F_n(\varphi)$ are unknown functions where $\varphi = \varphi(t)$. First of all, we list all field equations and possible conserved currents, then use the maps as follows:

$$\begin{array}{lll} 1.\ F_1(\varphi) \rightarrow F_1(t), \text{ So we have: } & & F_1'(\varphi) \rightarrow \frac{\dot{F}_1(t)}{\dot{\varphi}(t)}, \\ 2.\ F_2(\varphi) \rightarrow F_2(t), & & F_2'(\varphi) \rightarrow \frac{\dot{F}_2(t)}{\dot{\varphi}(t)}, \\ \vdots & \Longrightarrow & \vdots \\ n.\ F_n(\varphi) \rightarrow F_n(t), & & F_n'(\varphi) \rightarrow \frac{\dot{F}_n(t)}{\dot{\varphi}(t)}, \end{array}$$

where the prime indicates a derivative with respect to φ, and the dot indicates differentiation with respect to time. By substituting these in all equations, we may solve our ODE-system easily. After solving the system, we do an inverse map for obtaining the usual form of the unknown functions (i.e. depending on φ). Perhaps, in some cases, the inverse map be hard to obtain. In such cases, one can do it numerically. In numerical inverse mapping, only two options are in order: requiring initial values or the time interval. Note that one can first carry out a Noether approach for getting the conserved currents, and then proceed with this approach i.e. D.E-system = {field equations + Noether conserved currents + other conserved currents} without paying any attention to the form of the unknown functions which are obtained by the Noether approach. So, one could see that the form of the unknown function may be different from those derived from the Noether approach. Finding the solutions which carry all conserved currents found with the Noether approach is challenging but with the B.N.S.-approach this road is paved.

Now, we carry out this approach in our case.

For solving the field equations (Eqs. 6, 7 and 8), without any loss of generality, we use the maps as follows:

$$\begin{cases} i: \ f(\phi(t)) \longrightarrow f(t), \\ ii: V(\phi(t)) \longrightarrow V(t). \end{cases} \tag{53}$$

Therefore, we have

$$\begin{cases} i: \ f'(\phi) \longrightarrow \frac{\dot{f}(t)}{\dot{\phi}(t)}, \\ ii: V'(\phi) \longrightarrow \frac{\dot{V}(t)}{\dot{\phi}(t)}. \end{cases} \tag{54}$$

Let us add some other conserved currents from NS (Eq. 26). We would like to add $\mathbf{I}_2 = 0$, $\mathbf{I}_3 = 0$, and $\mathbf{I}_4 = c_0^2$, in which c_0 is a constant. Note that Eq. (9) (Hamiltonian constraint) is $\mathbf{I}_2$. Substituting Eqs. (53) and (54) in Eqs. (7), (8) and $\partial \mathbf{I}_4/\partial t = 0$, we obtain the modified system of differential equations

$$\begin{aligned} \mathbf{ODE-Sys.} = \{&\text{Eq. (6)} \cup \text{Eq. (7)} \cup \text{Eq. (8)} \cup \mathbf{I}_2 = 0 \\ &\cup \mathbf{I}_3 = 0 \cup d\mathbf{I}_4/\mathrm{d}t = 0\}. \end{aligned}$$

Solving it leads to (Set - 1)

$$\begin{aligned} \phi(t) &= \text{Any arbitrary function of time.} = b, \\ a(t) &= c_3 \exp\left(\frac{b}{\sqrt{6}}\right), \\ A(t) &= c_1 + c_2 \\ &\times \left(\int \left\{\exp\left[\int \left(\frac{3\sqrt{6}\dot{b}^3 + 2\sqrt{6}\dddot{b} + 18\dot{b}\ddot{b}}{6\dot{b}^2 + 2\sqrt{6}\ddot{b}}\right) \mathrm{d}t\right]\right\} \mathrm{d}t\right), \\ f(t) &= \frac{a\left(-2\left(3\dot{b}^2 + \sqrt{6}\ddot{b}\right)\right)^{1/2}}{2\dot{A}}, \\ V(t) &= \frac{1}{4}\left(\ddot{b} + 3\dot{b}^2\right). \end{aligned} \tag{55}$$

If we do calculations with $\mathbf{I}_4 = c_0^2$, instead of $\partial \mathbf{I}_4/\partial t = 0$ the results will be the same. Limpidly, we have three symmetry generators ($\mathbf{X}_2$, $\mathbf{X}_3$ and $\mathbf{X}_4$) with these solutions for the time being. We know that $\phi(t)$ must decay with time. According to Eq. (55), if we set a function with decaying nature for $\phi(t)$, it leads to the same behavior for $a(t)$ as well. So, it is not admissible, hence we desert this solution. There being more conserved currents is the reason for this non-physical solution.

Removing one of the conserved currents, $\mathbf{I}_3 = 0$, the results are (Set-2)

$$\begin{aligned} a(t) &= \text{Any arbitrary function of time.} = F_1, \\ A(t) &= \text{Any arbitrary function of time.} = F_2, \\ f(t) &= c_0 \left(F_1 \dot{F}_2\right)^{-1/2}, \\ \phi(t) &= c_1 + \int \left[\frac{\sqrt{-6 f^2 \dot{F}_2^2 - 18 F_1 \ddot{F}_1 + 18 \dot{F}_1^2}}{3F_1}\right] \mathrm{d}t, \\ V(t) &= \frac{-f^2 \dot{F_2}^2 + 6 F_1 \ddot{F}_1 + 12 \dot{F_1}^2}{6F_1^2} \end{aligned} \tag{56}$$

where $f = f(t)$. As we observe, we have degrees of freedom for choosing the form of $a(t)$ in both cases.

- An example for better understanding the analytical inverse map process.

In "**Set - 1**", one can assume the following non-physical form for b:

$$b = \sqrt{6}t^2 \quad \rightarrow \quad t^2 = \frac{\phi(t)}{\sqrt{6}}.$$

So, we obtain

$$a(t)=\exp(t^2),\quad A(t)=t\exp(3t^2),$$

$$f(t)=\frac{\sqrt{-6(6t^2+1)}\exp(-2t^2)}{6t^2+1}\longrightarrow f(\phi)=\frac{\sqrt{-6\left(\sqrt{6}\phi+1\right)}\exp\left(\frac{-2\phi}{\sqrt{6}}\right)}{\sqrt{6}\phi+1}\longrightarrow f(\phi)^2=\frac{-6\exp\left(\frac{-2\sqrt{6}\phi}{3}\right)}{\sqrt{6}\phi+1},$$

$$V(t)=18t^2+1\quad\longrightarrow\quad V(\phi)=\frac{18}{\sqrt{6}}\phi+3.$$

We proceed with the second case. We would like to do it for the "dark energy dominated era". For this purpose, we want to take the form of $\phi(t)$ and $A(t)$ arbitrary. However, we have room for choosing the form of $a(t)$, but we want it to arise from the heart of the equations spontaneously for comparing with the known scale factor for the dark energy dominated era (i.e. $a_{\text{D.E.}}=e^{-1}e^{H_0t}$. Here, the coefficient H_0 in the exponential, the Hubble constant, is about $7.25\times10^{-11}\,s^{-1}$, and the coefficient e^{-1} is for normalizing the scale factor to 1 at the present time). We take $\phi(t)$ with decreasing nature function as

$$\phi(t)=\frac{2.708908423\times10^5}{\sqrt{t}},\tag{57}$$

and the form of $A(t)$ as obtained from the NS-results (Eq. 34),

$$A(t)=-7.068617997\times10^{-14}t^{1/3}.\tag{58}$$

Solving Eq. (56) with Eqs. (57) and (58) numerically as a boundary value problem in the dark energy dominated era's time range, [9.8 Gyr, 13.8 Gyr], with these selections

$$c_0=1,\quad c_1=0,\quad a(t=9.8\times10^9)=0.7304711450,\quad a(t=13.8\times10^9)=1,\tag{59}$$

shows deep compatible results with observational data. We present Figs. 5 and 6 to demonstrate the results obtained. Figure 5a indicates the behaviors of the obtained scale factor, $a_{\text{B.N.S.}}(t)$, and the known scale factor for the dark energy dominated era, $a_{\text{D.E.}}(t)$ versus time. It shows good agreement between $a_{\text{B.N.S.}}(t)$ and $a_{\text{D.E.}}(t)$. However, maybe it is not a correct comparison, for we studied those versus time. For this reason, we study $a_{\text{B.N.S.}}(t)$ and $a_{\text{D.E.}}(t)$ versus their own redshifts. Plots overlapping in Fig. 5b show perfect agreement between $a_{\text{B.N.S.}}(t)$ and $a_{\text{D.E.}}(t)$ versus $z_{\text{B.N.S.}}$ and $z_{\text{D.E.}}$, respectively. Figure 6c shows the detractive behavior of the scalar field ϕ versus time, which leads to decreasing $V(\phi)$ and incremental $f^2(\phi)$ versus ϕ in Fig. 6a, b, respectively. Here, $f(\phi)$ is pure imaginary again, and as mentioned above, it is not problematic.

As an example, one can take $a(t)=c_1\sinh^{2/3}(c_2t)$, and describe the elaborations of cosmic evolution from matter dominated era till now.

5 WDW-equation

Let us proceed with Eq. (31). So, we have the following Hamiltonian:

$$\mathcal{H}=-\frac{3}{16}\Pi_u\Pi_w+4V_0u^2+2f_0^{-2}u^{-2/3}\Pi_A^2,\tag{60}$$

where $\{\Pi_j=\partial\mathcal{L}/\partial\dot{Q}^j\ \ ;Q^j\in\{w,u,v=A\}\}$ are the conjugated momenta of the configuration space. By a straight-

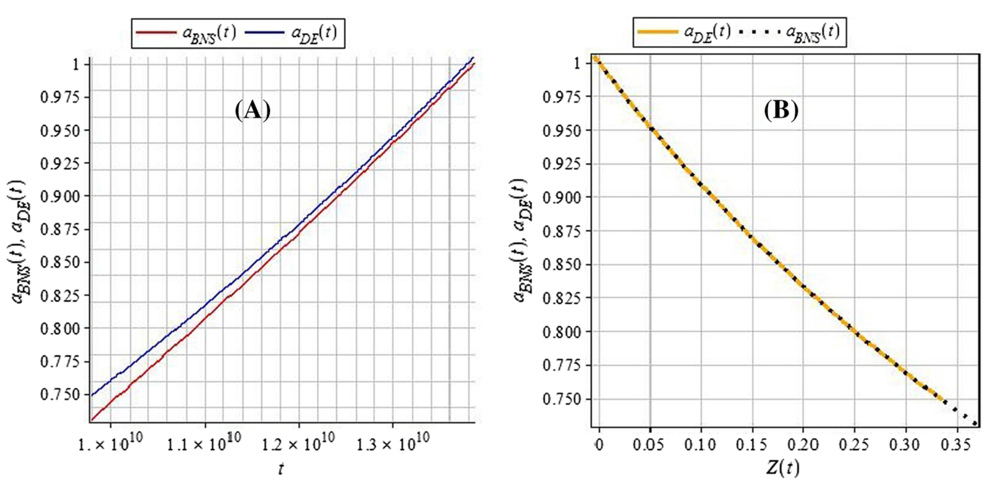

Fig. 5 *Plot* **a** indicates the scale factors $a_{\text{D.E.}}(t)$ and $a_{\text{B.N.S.}}(t)$ versus time t at the time range [9.8 Gyr, 13.8 Gyr], while **b** shows both scale factors versus their own redshifts z (i.e. $z_{\text{D.E.}}$ and $z_{\text{B.N.S.}}$) at the same time range evolving

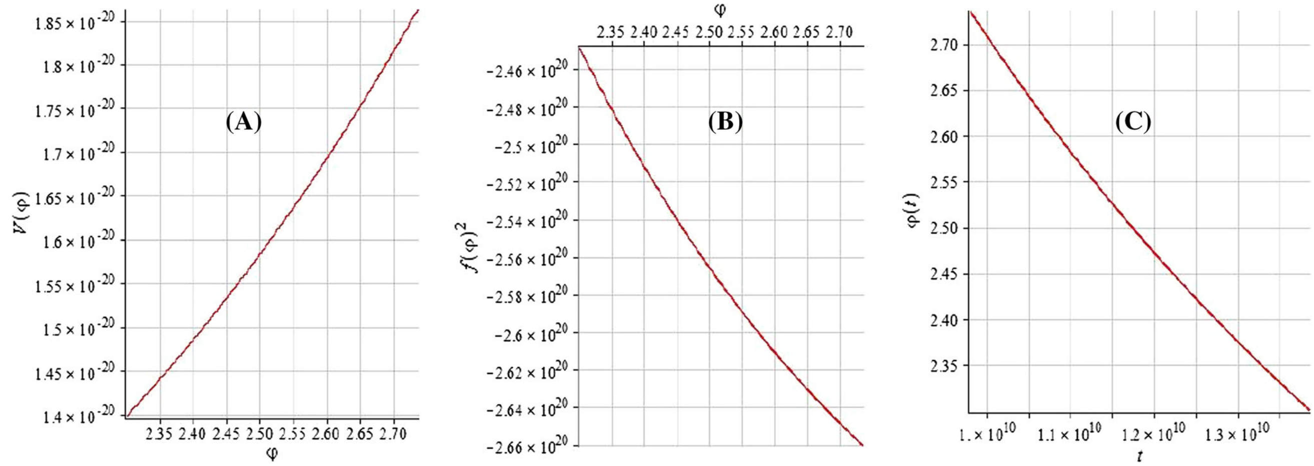

Fig. 6 *Plots* **a** and **b** indicate the scalar potential $V(\phi)$ and coupling function $f^2(\phi)$ versus scalar field ϕ at the time evolving range [9.8 Gyr, 13.8 Gyr], respectively, while *plot* **c** shows the scalar field ϕ versus time at the same time range

forward canonical quantization procedure, we have

$$\Pi_j \to \hat{\Pi}_j = -i\partial_j, \\ \mathcal{H} \to \hat{\mathcal{H}}(Q^j, -i\partial_{Q^j}). \tag{61}$$

The Hamiltonian constraint gives the Wheeler–De Witt equation,

$$\left[-\frac{3}{16}(-i\partial_u)(-i\partial_w) + 4V_0u^2 + 2f_0^{-2}u^{-2/3}(-i\partial_A)^2 \right] |\Psi(w, u, A)\rangle = 0, \tag{62}$$

in which $|\Psi(w, u, A)\rangle$ is the wave function of the universe. Pursuing the Noether symmetry, if we use the following two conserved currents:

$$\Pi_w = \Sigma_1, \quad \Pi_A = \Sigma_2, \tag{63}$$

then, according to [13],

$$|\Psi\rangle = \sum_{j=1}^{m} e^{i\Sigma_j Q^j} |\chi(Q^l)\rangle, \quad m < l \leq n, \tag{64}$$

where m is the number of symmetries, l are the directions where symmetries do not exist, n is the total dimension of the minisuperspace, and we have

$$|\Psi(w, u, A)\rangle = e^{i\Sigma_1 w} e^{i\Sigma_2 A} |\Theta(u)\rangle. \tag{65}$$

Note that the presence of the exponential functions is due to the separation of variables in Eq. (62) and the quantum version of the constraints (63), which are

$$-i\partial_w = \Sigma_1 |\Psi(w, u, A)\rangle, \quad -i\partial_A = \Sigma_2 |\Psi(w, u, A)\rangle.$$

After putting this solution (Eq. 65) in Eq. (62) and solving it, we get this perfect solution:

$$|\Psi(w, u, A)\rangle = c_1 e^{i\Sigma_1 w} e^{i\Sigma_2 A} e^{\frac{b_2 u^3 + 9b_3 u^{1/3}}{-3b_1}} \tag{66}$$

where $b_1 = 3i\Sigma_0/16$, $b_2 = 4V_0$, $b_3 = 2f_0^{-2}u^{-2/3}\Sigma_1^2$, and c_1 is an integration constant. It is clear that the oscillating feature of the wave function of the universe recovers the so-called Hartle criterion [53].

6 Conclusion

In this paper, we studied an original action in teleparallel gravity which has never been introduced in the literature. However, the $f(R)$ version of the action (1) has been introduced and studied in Refs. [45–48]. By the use of the Noether approach, we found that late-time-accelerated expansion is realized with this model. Our data analysis showed that the age of the universe is 13.86 Gyr, the present values of the scale factor, deceleration, and EoS parameters are $a_0 = 1.00$, $q_0 = -1$ and $W_{\text{eff}_0} = -1$, respectively, and the scalar field, ϕ, and the coupling function, $f^2(\phi)$, are of decreasing nature, while the scalar potential, $V(\phi)$, is increasing with time. Considering the deceleration parameter, we learned that the universe starts accelerating from $z = 0.524$, which is equivalent to $t_{ac.} = 6.294$ Gyr. The resulting model crosses the phantom divide line from the quintessence phase to the phantom phase. By data analysis, we obtained the quadratic coupling function, $f^2(\phi)$, to be real in the action (1) but $f(\phi)$ itself was pure imaginary. In other words, we found (by data analysis) that $f(\phi)$ has the form $f(\phi) = (\sqrt{-1})N(\phi) = iN(\phi) = f_0N(\phi)$ in which $N(\phi)$ is a real function of the scalar field ϕ i.e. $N(\phi) = \exp\left[\sqrt{6}\phi/(-12)\right]$ (see Eqs. 37 and 39), but it is not problematic and is compatible with observations, since according to the action (1), we see that f_0 appears with power 2 (i.e. f_0^2) in the action and also in the relevant equations (see Eqs. 6–11, 32, and 41–44). We showed that the obtained results satisfied Maxwell's equations in curved spacetime. The values of the electric and magnetic fields fall off with time and also they have the same norm at the present time. We presented an original approach, the "B.N.S. approach", for easily solving the field equations keeping almost all conserved currents which we want to have, and we showed that, unlike the Noether approach, we have solutions with this approach. Our solutions could describe the dark energy dominated era. In Sect. 5 we considered the WDW-equation and showed that the wave function of the universe has oscillating features which in the cosmic evolution recovers the so-called Hartle criterion.

Finally, we would like to compare our NS-results with SNS-results of Ref. [47], which considered an $f(R)$ version of the action (1). By taking the incorrect vector field as $A_\mu = (0; 0, 0, A(t))$, they showed that the scale factor is about $a(t) \sim (t^4 + t^{4/3} + t)^{1/3}$, while in our case it is about $a(t) \sim (t^4 + t^{4/3} + t + t^2)^{1/3}$. According to the extra term, t^2, in the parentheses, other things are different. However, regarding the FLRW spacetime, we took a different vector field; cf. (3). Anyway, both versions show late-time-accelerated expansion. It is worth to note that the behavior of the scalar field is different; that is, in our case, it is of a decreasing nature, while in that paper it is of an increasing nature after a little detractive behavior. In that paper, they studied the qualitative behavior of $a(t)$ and $\phi(t)$ versus time only, so we cannot discuss this more. Anyway, we are sure that in that paper, one can show the pure imaginary nature of $f(\phi)$ by data analysis, such as our case. However, we have $f^2(\phi)$ in that action, so it is not problematic.

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